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1994 J. Phys. A: Math. Gen. 27 2857

(http://iopscience.iop.org/0305-4470/27/8/022)

J. Phys. A: Math. Gen. 27 (1994) 2857–2878. Printed in the UK

Modified symmetry generators and the geometric phase

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Received 21 June 1993

Abstract. Coupled systems of slow and fast variables with symmetry, characterized by a semisimple Lie group G, are employed to study the effect of adiabatic decoupling of the fast degrees of freedom on the algebra of symmetry generators. The slow configuration space is assumed to be the symmetric coset space G/H, where H is a compact subgroup of G defined by the fast Hamiltonian. The induced gauge fields characterizing the effective slow dynamics are symmetric ones in the sense that the action of G on them can be compensated by an H-valued gauge transformation. The modification of the symmetry generators when such gauge fields are present can be described purely in geometric terms related to the non-Abelian geometric phase. The modified generators may be identified as the generators of the induced representation of G, where the inducing represention is the representation of H on the fast Hilbert space. This result enables us to recast the problem of exotic quantum numbers for effective quantum systems in purely algebraic terms via the Frobenius reciprocity theorem. Illustrative calculations for the symmetric spaces $SO(d+1)/SO(d) \sim S^d$ (spheres) are presented. Possible relevance of modified generators for non-compact G for obtaining scattering potentials in the framework of algebraic scattering theory is also pointed out.

1. Introduction

In physics we are often faced with the problem of describing the dynamics of coupled systems with different energy scales, e.g. systems involving slow and fast degrees of freedom. Examples of this kind ranging from the molecular Born–Oppenheimer approximation to field theory have been extensively studied by employing the concept of the geometric phase [1, 2]. It is now well known that the adiabatic decoupling of the fast variables from the slow ones may result in an effective theory with both Abelian and non-Abelian induced gauge fields. Such gauge fields have a crucial impact on the effective slow dynamics. The presence of such fields is responsible for the appearance of exotic quantum numbers, e.g. half-integer orbital angular momentum for diatoms [3, 4]. The main theme in these phenomena is that the good quantum numbers for the effective theory can be described as eigenvalues of some modified set of operators commuting with the effective Hamiltonian.

It is well known that the algebra of symmetry generators has to be modified when gauge-fields are present [5, 6]. This idea was emphasized in [7] in connection with the geometric phase. The crucial observation for applying the procedure of [5] and [6] is that the gauge fields usually appearing in the examples concerning the geometric phase are symmetric ones, in the sense that the effect of a symmetry transformation on the fields can be compensated by a suitable gauge transformation. This point was further emphasized

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by Vinet [8] in his differential geometric treatment of symmetric gauge fields and the geometric phase. However in [8] the dynamics of only the fast variables was considered, and the parameters of the Hamiltonian played no dynamical role. In this paper, by promoting the external parameters to slow dynamical variables, we examine the geometric origin of modified symmetry generators. In order to do this we consider a simple class of models where the slow configuration space is a coset space G/H. We impose further restrictions on our class of models so as to enable an explicit construction of the modified symmetry generators. The first restriction is that G/H is a symmetric space. The second is that the matrix elements, with respect to a degenerate eigensubspace of the fast Hamiltonian, of the generators of g *not belonging* to h are vanishing (g and h are the Lie algebras of G and H).

In section 2 we present our class of models and discuss their basic geometrical properties. In section 3 we show that symmetric gauge fields arise naturally for such models. Using some elementary properties of symmetric coset spaces, in section 4 we explicitly construct the Killing vectors (generators of the infinitesimal action of G on G/H), and compensating gauge transformations in terms of the structure constants of g and the normal coordinates on G/H. In section 5 we discuss the symmetry properties of our models. Employing the Born–Oppenheimer method we obtain an effective slow Hamiltonian. We also define here the modified symmetry generators as the ones commuting with the effective Hamiltonian. In section 6.1 we illustrate our results for a class of models where the slow configuration spaces are spheres $S^d \sim SO(d+1)/SO(d)$. For the special case of $d = 2$ we obtain the well known modification of the angular momentum algebra when a magnetic monopole is present [7]. For $n = 4$ we get a modified set of $SO(5)$ generators used, for example, by Yang for the construction of $SU(2)$ monopole harmonics [9]. In section 6.2 it is argued that realizations of non-compact groups in terms of modified symmetry generators may be useful for obtaining analytical expressions for interaction terms corresponding to scattering problems, in the spirit of algebraic scattering theory. In section 7 we show that the appearence of exotic quantum numbers in effective quantum systems can be described using the theory of induced representations. Our conclusions are left for section 8.

2. Coset space models

Let G be a semisimple Lie-group and H a compact subgroup of G. We denote by g and h the Lie algebras of G and H, both are spanned by anti-Hermitian generators. The Cartan–Killing metric

$$\eta_{IJ} = C_{IK}{}^{L} C_{JL}{}^{K} \qquad I, J, \ldots = 1, 2, \ldots, \dim g \tag{2.1}$$

is non-degenerate. $C_{IJ}{}^{K}$ are the structure constants of g. For G compact the metric can be taken to be proportional to the identity.

The class of models of coupled slow and fast variables is defined by the total Hamiltonian

$$H_{\mathrm{tot}} = -\frac{1}{2M}\eta^{IJ}\mathcal{K}_I\mathcal{K}_J + \mathcal{U}^{(\lambda)}(x)H_0\mathcal{U}^{(\lambda)\dagger}(x) \equiv H_{\mathrm{kin}} + H(x) \tag{2.2}$$

where $x \in G/H$ are coordinates on the symmetric coset space, $\mathcal{K}_I$ are the Killing vectors generating infinitesimal G rotations on G/H (described below), and $\mathcal{U}^{(\lambda)}$ is a unitary irreducible representation of G labelled by λ. Moreover H_0 does not depend on x, and commutes with the restriction of $\mathcal{U}^{(\lambda)}$ to the subgroup H. Recall also that $\mathcal{K}_I$ are first-order differential operators, and the combination $\eta^{IJ}\mathcal{K}_I\mathcal{K}_J$ (η^{IJ} is the inverse of η_{IJ}) is the second-order Laplace–Beltrami operator on G/H. It is clear that the slow configuration space is G/H and $H_{\rm kin}$ corresponds to the kinetic energy for a particle with mass M constrained to move on G/H. We assume that M is sufficiently large in order to ensure the validity of the Born–Oppenheimer approximation.

In order to clarify the meaning of our class of models, let us consider a specific example. Let us choose $G = SU(2)$, $H = U(1)$; hence our coset is $S^2 \sim SU(2)/U(1)$, the 2-sphere. We can parametrize S^2 (locally) by the polar coordinates (θ, φ). However, let us choose instead of these the coordinates (x_1, x_2), where

$$x_1 = -\theta \sin\varphi \qquad x_2 = \theta\cos\varphi \,. \tag{2.3}$$

We can parametrize an element of $SU(2)$, by the coset space coordinates x_1, x_2 in the form

$$L(x) \equiv \mathrm{e}^{x_1\mathcal{J}_1 + x_2\mathcal{J}_2} \in SU(2) \tag{2.4}$$

where $\mathcal{J}_1$ and $\mathcal{J}_2$ are the generators of the Lie-algebra $su(2)$ *not belonging* to the subalgebra $u(1)$ taken to be spanned by the generator $\mathcal{J}_3$. Of course the (anti-Hermitian) generators $\mathcal{J}_I$ $(I = 1, 2, 3)$ satisfy

$$[\mathcal{J}_I, \mathcal{J}_J] = \varepsilon_{IJ}{}^K \mathcal{J}_K \qquad I, J, K = 1, 2, 3 \,. \tag{2.5}$$

Let us restrict our attention to a particular representation $\mathcal{U}^{(\lambda)}$ of $SU(2)$. In particular we can represent the group element (2.4) in the form

$$\mathcal{U}^{(\lambda)}(x) \equiv \mathcal{U}^{(\lambda)}(L(x)) = \mathrm{e}^{x_1\mathcal{J}_1 + x_2\mathcal{J}_2} \tag{2.6}$$

where for simplicity we have used the same letters $\mathcal{J}_I$ for the generators of $su(2)$ in the representation $\mathcal{U}^{(\lambda)}$. Choosing H_0 to be $\mathcal{J}_3$, then using (2.3) and (2.6) one can show that

$$\mathcal{U}^{(\lambda)}(x) H_0 \mathcal{U}^{(\lambda)\dagger}(x) = \mathcal{J}_I X_I \equiv H(\boldsymbol{X}) \tag{2.7}$$

where $(X_1, X_2, X_3) \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. The Killing vectors generating the infinitesimal *left action* of $SU(2)$ on S^2 are

$$\mathcal{K}_I = \varepsilon_{IJK} X_J \frac{\partial}{\partial X^K} \tag{2.8}$$

i.e. the usual set of (anti-Hermitian) orbital angular momentum operators satisfying [10] $[\mathcal{K}_I, \mathcal{K}_J] = -\varepsilon_{IJK}\mathcal{K}_K$.

As $H_{\rm kin}$ is the angular momentum part of the standard Laplacian in $\boldsymbol{R}^3$, we see that it generates motion on the sphere. Using $H_{\rm kin}$ together with $H(\boldsymbol{X})$ of (2.7) we recover the well known example of a spin coupled to an outward magnetic field. The unit vector $\boldsymbol{X}$ which determines the direction of the magnetic field, plays a dynamical role due to the presence of $H_{\rm kin}$ in the total Hamiltonian.

Let us now explore the geometrical properties of our models. Recall first that a *left coset* is the set of equivalence classes of the form gH where $g \in G$. Two elements $g_1, g_2 \in G$ belong to the same equivalence class G/H iff there exists $h \in H$ such that $g_2 = g_1 h$. Moreover there is a natural *left action* of G on G/H defined for $g \in G$ by the mapping $(g_1 H) \to g(g_1 H) = (g g_1 H)$. For practical purposes it is convenient to introduce some (local) coordinates x^μ $(\mu = 1, 2, \ldots, \dim G/H)$ on G/H, and an explicit mapping $x \mapsto gx$.

Especially we will be interested in the infinitesimal form of this mapping with $g \sim e + u$ i.e. $x \mapsto x + \delta x(u, x)$. The explicit form of $\delta x(u, x)$ will be given in section 4.

As a next step we split the Lie algebra g according to $g = h \oplus m$. Since our Lie-algebra is semi-simple the subspace m can be regarded as the orthogonal complement of h with respect to the Cartan–Killing metric. Let us denote the generators of h by S_a ($a = 1, 2, \ldots, \dim h$) and the generators belonging to m by T_α ($\alpha = 1, 2, \ldots, \dim g - \dim h$). Hence

$$\{\mathcal{J}_I\} \equiv \{S_a\} \oplus \{T_\alpha\} . \tag{2.9}$$

m is actually the tangent space to G/H at the identity coset therefore we can use the generators $T_\alpha \in m$ to parametrize a group element (coset representative) in the form

$$L(x) = \mathrm{e}^{x^\mu \delta_\mu{}^\alpha T_\alpha} \in G \qquad \mu, \alpha = 1, 2, \ldots, \dim G/H . \tag{2.10}$$

Notice that the indices $(\alpha, \beta, \ldots)$ refer to the properties of G/H at the identity coset, while the indices $(\mu, \nu, \ldots)$ with the same range refer to the coordinates x^μ at a neighbourhood of the identity (normal coordinates). However, for simplicity we shall omit the $\delta_\mu{}^\alpha$ in cases where no confusion arises and use the greek indices $(\alpha, \beta, \ldots)$ and $(\mu, \nu, \ldots)$ interchangeably.

As G/H being a symmetric space we have the following set of commutation relations [11]:

$$[S_a, S_b] = C_{ab}{}^c S_c \qquad [S_a, T_\alpha] = C_{a\alpha}{}^\beta T_\beta \qquad [T_\alpha, T_\beta] = C_{\alpha\beta}{}^a S_a . \tag{2.11}$$

Now we discuss the symmetry properties of the eigensubspaces of $H(x)$ of (2.2). Let us denote the identity coset by x_0 which can be regarded as a reference point. We also assume that the eigenvalues and eigenvectors of $H_0 \equiv H(x_0)$ are known

$$H_0|n, j(x_0)\rangle = \varepsilon^{(n)}|n, j(x_0)\rangle . \tag{2.12}$$

The dimension of the Hilbert space $\mathcal{H}$ is fixed by the representation $\mathcal{U}^{(\lambda)}$, $j = 1, 2, \ldots, \dim\mathcal{H}_n$, $\mathcal{H}_n$ is the eigensubspace of $\mathcal{H}$ corresponding to the eigenvalue $\varepsilon^{(n)}$. Moreover, since H_0 commutes with the restriction of $\mathcal{U}^{(\lambda)}$ to the subgroup H we can define

$$\mathcal{U}^{(\lambda)}(h)|n, j(x_0)\rangle \equiv |n, i(x_0)\rangle \mathcal{R}_{ij}{}^{(n)}(h) \qquad h \in H \tag{2.13}$$

where we assume that $\mathcal{R}^{(n)}$ is a unitary irreducible representation of H occuring in the restriction of $\mathcal{U}^{(\lambda)}$ to H (which is generally reducible).

The eigenvectors of $H(x)$ have the following form:

$$|n, j(x)\rangle = \mathcal{U}^{(\lambda)}(L(x))|n, j(x_0)\rangle \tag{2.14}$$

where we used the definition of the coset representative (2.10) and (2.2). How does $\mathcal{U}^{(\lambda)}(g)$ $g \in G$ act on $|n, j(x)\rangle$? We expect

$$\mathcal{U}^{(\lambda)}(g)|n, j(x)\rangle = \mathcal{U}^{(\lambda)}(gL(x))|n, j(x_0)\rangle = |n, j(gx)\rangle . \tag{2.15}$$

However, $|n, j(gx)\rangle$ remains an eigenvector of $H(gx)$ corresponding to $\varepsilon^{(n)}$ if we make a transformation

$$|n, j(gx)\rangle \mapsto |n, i(gx)\rangle \mathcal{R}_{ij}{}^{(n)}(h(g, x)) \tag{2.16}$$

where we can also allow an x dependence for $h \in H$. The final result for the action of the symmetry group G on the eigensubspaces of $H(x)$ is

$$\mathcal{U}^{(\lambda)}(g)|n, j(x)\rangle = |n, i(gx)\rangle \mathcal{R}_{ij}{}^{(n)}(h(g,x)). \tag{2.17}$$

This property of the eigensubspaces can be traced back to the property of the coset G/H

$$gL(x) = L(gx)h(g,x) \tag{2.18}$$

by virtue of (2.13), (2.14) and (2.17). Since $h(g,x) \in H$ we can write

$$h(g,x) \equiv \mathrm{e}^{y^a(g,x)S_a} \equiv \mathrm{e}^{y(g,x)S}. \tag{2.19}$$

Using (2.10) and adopting the notation $x^\mu \delta_\mu{}^\alpha T_\alpha \equiv xT$ we can write (2.18) in the following form:

$$g\,\mathrm{e}^{xT} = \mathrm{e}^{x'(g,x)T}\,\mathrm{e}^{y(g,x)S}. \tag{2.20}$$

3. Symmetric gauge fields

Now we show that as a result of (2.20) symmetric gauge fields will arise. Let us introduce $\mathrm{d} \equiv \mathrm{d}x^\mu \partial_\mu$. Regarding g as fixed we differentiate (2.20) and eliminate g to obtain

$$\mathrm{e}^{-x'T}\,\mathrm{d}\,\mathrm{e}^{x'T} = \mathrm{e}^{yS}\,(\mathrm{e}^{-xT}\,\mathrm{d}\,\mathrm{e}^{xT} + \mathrm{d})\,\mathrm{e}^{-yS}. \tag{3.1}$$

As a next step we would like to know the infinitesimal version of (3.1). Let $g = e + u \equiv e + u^a S_a + u^\alpha T_\alpha$, u^I $(I = 1, 2, \ldots, \dim g)$ is infinitesimal. Then

$$x'^\mu = x^\mu + \delta x^\mu(u,x) \equiv x^\mu + u^I \delta x_I{}^\mu(x) \tag{3.2a}$$

$$y^a \equiv \delta y^a(u,x) \equiv u^I \delta y_I{}^a(x). \tag{3.2b}$$

Using the notation

$$\omega = \omega_\mu(x)\,\mathrm{d}x^\mu \equiv L^{-1}(x)\,\mathrm{d}L(x) = \mathrm{e}^{-xT}\,\mathrm{d}\mathrm{e}^{xT} \tag{3.3}$$

we obtain for the left-hand side of (3.1)

$$\omega_\mu(x) + \delta x^\nu(u,x)\partial_\nu \omega_\mu + \partial_\mu(\delta x^\nu(u,x))\omega_\nu(x) \tag{3.4}$$

and for the right-hand side of (3.1)

$$\omega_\mu(x) + [\delta y(u,x)S, \omega_\mu(x)] - \partial_\mu(\delta y(u,x)S). \tag{3.5}$$

The second and third terms in (3.4) define the Lie derivative of $\omega_\mu(x)$ in the direction defined by $\delta x^\mu(u,x)$, while the corresponding terms of (3.5) define *minus* the covariant derivative $\nabla^{(\omega)}{}_\mu$ of $\delta y(u,x)S$ with respect to the gauge field $\omega_\mu(x)$.

Introducing the notation [6]

$$f^\mu(u,x) \equiv u^I f_I{}^\mu(x) \equiv -u^I \delta x_I{}^\mu(x) \tag{3.6a}$$

$$W_f(u,x) \equiv u^I W_I(x) \equiv u^I \delta y_I{}^a(x) S_a \tag{3.6b}$$

$$\nabla_\mu{}^{(\omega)} \equiv \partial_\mu + [\omega_\mu, .] \tag{3.6c}$$

the infinitesimal version of (3.1) can be recast in the form

$$\mathcal{L}_f \omega_\mu = \nabla_\mu{}^{(\omega)} W_f \tag{3.7}$$

which is the symmetry condition for ω_μ according to [5] and [6].

Now we can see that by solving (2.20) for the infinitesimal quantities defined by (3.2*a*, *b*) we can obtain the explicit form of the coordinate transformation $x \mapsto x + \delta(u,x)$ corresponding to the group action of G on G/H. Moreover we realize that ω_μ is a symmetric gauge field in the sense that its response to this infinitesimal coordinate transformation ($\mathcal{L}_f \omega_\mu$) can be compensated by an infinitesimal gauge transformation defined by W_f of (3.6*b*).

Since $\omega_\mu \mathrm{d}x^\mu$ is a $\boldsymbol{g}$-valued 1-form we can expand it in terms of the generators belonging to the subspaces $\boldsymbol{h}$ and $\boldsymbol{m}$,

$$\omega_\mu \, \mathrm{d}x^\mu \equiv A_\mu{}^a \, \mathrm{d}x^\mu S_a + E_\mu{}^\alpha \mathrm{d}x^\mu T_\alpha \equiv A_\mu \, \mathrm{d}x^\mu + E_\mu \, \mathrm{d}x^\mu \,. \tag{3.8}$$

The curvature 2-form of ω is $\Omega = \mathrm{d}\omega + \omega \wedge \omega = 0$ hence the symmetric gauge-field ω_μ is trivial. *However after separating the* $\boldsymbol{h}$ *and* $\boldsymbol{m}$ *parts of* (3.7) *we get*

$$\mathcal{L}_f A_\mu = \nabla_\mu W_f \equiv \partial_\mu W_f + [A_\mu, W_f] \tag{3.9a}$$

$$\mathcal{L}_f E_\mu = [E_\mu, W_f]\,. \tag{3.9b}$$

Moreover the curvature of A, $F = \mathrm{d}A + A \wedge A = -E \wedge E$. Hence the $\boldsymbol{h}$ component of ω is a non-trivial symmetric gauge field.

Now let us recall that such results arose by virtue of (2.18) characterizing the geometry of our coset space G/H. However,we are interested in the structure of the eigensubspaces over G/H encoded in (2.17). (In mathematical terms we are interested in the structure of the homogeneous vector bundle associated with the canonical principal bundle G over G/H.) Since the induced gauge fields, relevant to the dynamics have the form [1,2]

$$\begin{aligned} \mathcal{A}_{\mu ij}{}^{(n)} &= \langle n, i(x)|\partial_\mu|n, j(x)\rangle = \langle i(x_0)|\omega_\mu|n, j(x_0)\rangle \\ &= \langle n, i(x_0)|S_a|n, j(x_0)\rangle A_\mu{}^a + \langle n, i(x_0)|T_\alpha|n, j(x_0)\rangle E_\mu{}^\alpha \end{aligned} \tag{3.10}$$

we have to calculate the matrix elements of the generators S_a and T_α in the base spanned by the vectors in the subspace $\mathcal{H}_n$. These matrix elements provided by the irreducible representation defined by (2.13) will be denoted by $S_a{}^{(n)}$ and $T_\alpha{}^{(n)}$ [11]. They are actually $\dim \mathcal{H}_n \times \dim \mathcal{H}_n$ matrices. In this paper we will only consider models where the matrix elements $T_\alpha{}^{(n)}$ are zero. Such restriction was employed [11] to show that in this case $\mathcal{A}_\mu{}^{(n)}$ corresponds to the connection obtained by lifting the natural Riemannian connection on G/H to our homogeneous vector bundle with fibres the degenerate eigensubspaces. Hence the non-trivial symmetric gauge fields that characterize the dynamics governed by *the Hamiltonian* $H(x)$ *appearing in* (2.2) *are of the form*

$$\mathcal{A}_{\mu ij}{}^{(n)}(x) = A_\mu{}^a(x)(S_a{}^{(n)})_{ij}\,. \tag{3.11}$$

Due to the non-triviality of $\mathcal{A}_\mu{}^{(n)}$ the non-Abelian version of Berry's phase appears, and it cannot be globally transformed away. Hence such gauge-fields will have an impact on the dynamics of the parameters x when we add to $H(x)$ the kinetic term H_{kin} of (2.2) and adiabatically decouple the fast degrees of freedom using the Born–Oppenheimer approximation. As a result, the modification of the dynamics will yield modified symmetry generators for the group G, commuting with the effective Hamiltonian of the slow system.

4. Killing vectors and compensating gauge transformations

In this section we solve the infinitesimal version of (2.20) for the quantities $\delta x_I{}^\mu$ and $\delta y_I{}^a$ defined by (3.2*a*, *b*). The Killing vectors generating the *left action* of G on G/H are the differential operators

$$\mathcal{K}_I \equiv \delta x_I{}^\mu \partial_\mu \qquad I = 1, 2, \ldots, \dim g \tag{4.1}$$

satisfying [10]

$$[\mathcal{K}_I, \mathcal{K}_J] = -C_{IJ}{}^K \mathcal{K}_K \qquad I, J, K = 1, 2, \ldots, \dim g \tag{4.2}$$

and the compensating gauge transformations W_I are defined by (3.6*b*).

Let us first define the following quantities

$$L^{-1}(x) S_a L(x) \equiv \mathcal{D}_a{}^b(x) S_b + \mathcal{D}_a{}^\beta(x) T_\beta \tag{4.3a}$$

$$L^{-1}(x) T_\alpha L(x) \equiv \mathcal{D}_\alpha{}^b(x) S_b + \mathcal{D}_\alpha{}^\beta(x) T_\beta \tag{4.3b}$$

with $L(x)$ defined by (2.10). Recall that the mapping $\mathcal{J}_I \mapsto L(x)\mathcal{J}_I L^{-1}(x) = (\mathcal{D}^{-1})_I{}^J \mathcal{J}_J$ defines the adjoint action of $L(x)$ on g. Being an automorphism of g, this mapping leaves the Cartan–Killing metric invariant, i.e.

$$\eta_{IJ} \equiv (\mathcal{J}_I, \mathcal{J}_J) = ((\mathcal{D}^{-1})_I{}^K \mathcal{J}_K, (\mathcal{D}^{-1})_J{}^L \mathcal{J}_L) = (\mathcal{D}^{-1})_I{}^K (\mathcal{D}^{-1})_J{}^L \eta_{KL}\,. \tag{4.4}$$

Using (2.18), (2.20) and (3.2*a*, *b*) with $g = e + u^I \mathcal{J}_I$ we obtain

$$L(x + \delta x) = (e + u^a S_a + u^\alpha T_\alpha) L(x)(e - \delta y^a S_a)\,. \tag{4.5}$$

After multiplication by $L^{-1}(x)$ we get by virtue of (3.3) and (3.8)

$$u^I \delta x_I{}^\mu (A_\mu{}^a S_a + E_\mu{}^\alpha T_\alpha) = u^I (\mathcal{D}_I{}^a S_a + \mathcal{D}_I{}^\alpha T_\alpha) - u^I \delta y_I{}^a S_a\,. \tag{4.6}$$

Separating terms proportional to S_a and T_α we get,

$$\delta x_I{}^\mu = \mathcal{D}_I{}^\alpha E_\alpha{}^\mu \tag{4.7a}$$

$$\delta y_I{}^a = \mathcal{D}_I{}^a - \mathcal{D}_I{}^\alpha \Pi_\alpha{}^a \tag{4.7b}$$

where

$$\Pi_\alpha{}^a \equiv E_\alpha{}^\mu A_\mu{}^a\,. \tag{4.8}$$

Notice that $E_\mu{}^\alpha$ is a quadratic non-singular matrix and in (4.7*a*, *b*) its inverse has been used. We can now express the Killing vectors as

$$\mathcal{K}_I = \mathcal{D}_I{}^\alpha E_\alpha{}^\mu \partial_\mu \tag{4.9}$$

and multiplication of the 'vector' S_a by the matrix $\delta y_I{}^a$ of (4.7*b*) gives the compensating gauge transformation.

Now we give explicit formulae for these quantities exploiting the (2.10) form of $L(x)$. Using the formula $e^{A}B\,e^{-A} = B+[A,B]+\frac{1}{2}[A,[A,B]]+\cdots$ and the commutation relations in (2.11) we obtain

$$\mathcal{D}_I{}^J = \begin{pmatrix} \mathcal{D}_a{}^b & \mathcal{D}_a{}^\alpha \\ \mathcal{D}_\alpha{}^a & \mathcal{D}_\alpha{}^\beta \end{pmatrix} = \begin{pmatrix} \cos\sqrt{NM} & N(\sin(\sqrt{MN}/\sqrt{MN})) \\ -(\sin(\sqrt{MN}/\sqrt{MN}))M & \cos\sqrt{MN} \end{pmatrix} \tag{4.10}$$

where

$$N_a{}^\alpha = -x^\mu C_{\mu a}{}^\alpha \tag{4.11a}$$

$$M_\alpha{}^a = x^\mu C_{\mu\alpha}{}^a \,. \tag{4.11b}$$

We also need explicit expressions for $A_\mu{}^a$ and $E_\mu{}^\alpha$. Assuming B small, we use the formula [12]

$$e^{A+B} = e^{A}\left(1+\int_0^1 e^{-tA}B\,e^{tA}\,dt\right)+\cdots \tag{4.12}$$

with $A \equiv xT$, $B \equiv dxT$ to obtain

$$A_\mu{}^a = -\delta_\mu{}^\alpha\int_0^1 dt\left(\frac{\sin\sqrt{MN}t}{\sqrt{MN}}M\right)_\alpha^{\ a} = -\delta_\mu{}^\alpha\int_0^1 dt\left(M\frac{\sin\sqrt{NM}t}{\sqrt{NM}}\right)_\alpha^{\ a} \tag{4.13a}$$

$$E_\mu{}^\alpha = \delta_\mu{}^\beta\int_0^1 dt\left(\cos\sqrt{MN}t\right)_\beta^{\ \alpha}. \tag{4.13b}$$

Note, that in these formulae the functions of the matrices NM and MN are computed using the power-series expansions

$$\cos\sqrt{MN} = I-\frac{1}{2!}(MN)+\frac{1}{4!}(MN)^2-\cdots \tag{4.14a}$$

$$\frac{\sin\sqrt{MN}}{\sqrt{MN}} = I-\frac{1}{3!}(MN)+\frac{1}{5!}(MN)^2-\cdots \tag{4.14b}$$

and in (4.13a) the formula $Mf(NM) = f(MN)M$ is used which is valid for analytic functions [13].

One can easily prove using the above calculated quantities that

$$\delta x_a{}^\mu = N_a{}^\mu \tag{4.15a}$$

$$\delta y_a{}^b = \delta_a{}^b \tag{4.15b}$$

hence the $\boldsymbol{h}$ components of the Killing vectors and compensating gauge transformations are

$$\mathcal{K}_a = N_a{}^\mu\partial_\mu \tag{4.16a}$$

$$W_a = S_a\,. \tag{4.16b}$$

For the $\boldsymbol{m}$ components $(\mathcal{K}_\alpha, W_\alpha)$ we obtain more complicated expressions which are nonlinear in x^μ. We will calculate these components using an explicit example in section 6.

5. Modified symmetry generators

As a next step we discuss the symmetry properties of the *total* Hamiltonian of (2.2). We can employ two types of G transformations, a transformation of the form $x^\mu \mapsto x^\mu + \delta x^\mu(u, x)$ on the slow (coset) variables, and one of the form $|n, j(x)\rangle \mapsto \mathcal{U}^{(\lambda)}(g)|n, j(x)\rangle$ on the fast ones, where $g = e + u$, u infinitesimal. One can easily show that a combined infinitesimal rotation of both types of variables leaves the system invariant.

Let us denote by $\mathcal{J}_I \quad (I = 1, 2, \ldots, \dim \mathfrak{g})$ the infinitesimal generators of G in the representation defined by $\mathcal{U}^{(\lambda)}$, hence $\mathcal{U}^{(\lambda)}(g) \sim I + u^I \mathcal{J}_I$. $\mathcal{J}_I$ are the generators of infinitesimal *fast* rotations. Similarly the vector fields

$$G_I \equiv -\delta x_I{}^\mu(x)\partial_\mu \tag{5.1}$$

related to the Killing vector fields by

$$\mathcal{K}_I = -G_I \tag{5.2}$$

generate infinitesimal *slow* rotations. (Recall the presence of the minus sign [10] also present in (3.6a) and (4.2).) Both sets of generators satisfy the commutation relations of $\mathfrak{g}$ i.e.

$$[\mathcal{J}_I, \mathcal{J}_J] = C_{IJ}{}^K \mathcal{J}_K \tag{5.3a}$$

$$[G_I, G_J] = C_{IJ}{}^K G_K\,. \tag{5.3b}$$

Using (2.17) one can see that

$$\mathcal{U}^{(\lambda)}(g)H(x)\mathcal{U}^{(\lambda)\dagger}(g) = H(gx)\,. \tag{5.4}$$

Moreover, employing the infinitesimal transformations $gx = x + \delta x(u, x)$, $\mathcal{U}^{(\lambda)}(g) \sim I + u^I \mathcal{J}_I$ we get, using (5.1) and (5.4),

$$[\mathcal{J}_I + G_I, H(x)] = 0\,. \tag{5.5}$$

As the kinetic term is the second-order Laplace–Beltrami operator $[G_I, H_{\rm kin}] = 0$, we conclude that

$$[\mathcal{J}_I + G_I, H_{\rm tot}] = 0\,. \tag{5.6}$$

As a next step we employ the Born–Oppenheimer method to obtain an effective slow Hamiltonian. We merely refer to the result which is a straightforward generalization of the one presented in [2, 14]

$$H_{\rm eff}{}^{(n)} = -\frac{1}{2M}\eta^{IJ}\mathcal{B}_I{}^{(n)}\mathcal{B}_J{}^{(n)} + V^{(n)} + \varepsilon^{(n)} \tag{5.7}$$

where

$$(\mathcal{B}_I{}^{(n)})_{ij} = -\delta x_I{}^\mu(\delta_{ij}\partial_\mu + \mathcal{A}_{\mu ij}{}^{(n)}) \equiv -\delta x_I{}^\mu(\nabla_\mu{}^{(n)})_{ij} \tag{5.8}$$

with $\mathcal{A}_{\mu ij}{}^{(n)}$ defined by (3.11) and

$$(V^{(n)})_{ij} \equiv -\frac{1}{2M}\eta^{IJ}\langle n, i(x)|\mathcal{K}_I|m, l(x)\rangle\langle m, l(x)|\mathcal{K}_J|n, j(x)\rangle \qquad m \neq n\,. \tag{5.9}$$

In (5.9) the range of I varies according to the dimension of the corresponding eigensubspace $\mathcal{H}_m$ with energy $\varepsilon^{(m)}$. (Summation for the repeated indices is implicit.) The terms $\mathcal{A}_\mu{}^{(n)}$ and $V^{(n)}$ are responsible for the appearance of the gauge forces of magnetic and electric type [14, 15].

Now we have only slow variables hence we expect some modification of the slow generator of (5.1), say $G_I{}^{(n)}$, satisfying

$$[G_I{}^{(n)}, H_{\mathrm{eff}}{}^{(n)}] = 0 . \tag{5.10}$$

The obvious guess for $G_I{}^{(n)}$ by replacing ∂_μ in (5.1) by the covariant derivative $\nabla_\mu{}^{(n)}$ hence obtaining $\mathcal{B}_I{}^{(n)}$ of (5.8), fails to close under commutation to a Lie algebra of G since

$$[\mathcal{B}_I{}^{(n)}, \mathcal{B}_J{}^{(n)}] = C_{IJ}{}^K \mathcal{B}_K{}^{(n)} + \delta x_I{}^\mu \delta x_J{}^\nu F_{\mu\nu}{}^{(n)} \tag{5.11}$$

as can easily verified using $\delta x_I{}^\nu \partial_\nu(\delta x_J{}^\mu) - \delta x_J{}^\nu \partial_\nu(\delta x_I{}^\mu) = -C_{IJ}{}^K \delta x_K{}^\mu$ which is equivalent to the commutation relation for the Killing vectors of (4.2).

Now we exploit the symmetry property (3.9a) also satisfied by $\mathcal{A}_\mu{}^{(n)}$ of (3.11) with

$$W_f{}^{(n)}(u, x) \equiv u^I W_I{}^{(n)}(x) \equiv u^I \delta y_I{}^a(x) S_a{}^{(n)} \tag{5.12}$$

to construct the correct set of modified generators $G_I{}^{(n)}$ satisfying the commutation relations of g. First we refer to the result [5, 6]

$$\mathcal{L}_{f_I} W_J{}^{(n)} - \mathcal{L}_{f_J} W_I{}^{(n)} + [W_I{}^{(n)}, W_J{}^{(n)}] = C_{IJ}{}^K W_K{}^{(n)} \tag{5.13}$$

which is a consistency condition for the compensating gauge transformation, and we used the definition (3.6a). Using (5.13) one can prove that the modified set of generators

$$G_I{}^{(n)} \equiv G_I + W_I{}^{(n)} = -\delta x_I{}^\mu \partial_\mu + W_I{}^a S_a{}^{(n)} \tag{5.14}$$

satisfy the commutation relations of g. An alternative form of $G_I{}^{(n)}$ can be obtained by introducing the quantity [5, 6]

$$\Phi_I{}^{(n)}(x) \equiv \Phi_I{}^a(x) S_a{}^{(n)} \equiv W_I{}^{(n)}(x) + \delta x_I{}^\mu(x) \mathcal{A}_\mu{}^{(n)}(x) . \tag{5.15}$$

Expressed in terms of $\Phi_I{}^{(n)}$

$$G_I{}^{(n)} = \mathcal{B}_I{}^{(n)} + \Phi_I{}^{(n)} \tag{5.16}$$

which is a modification of the term we tried to use in (5.11). The quantity $\Phi_I{}^{(n)}$ needed to cancel the extra term on the right-hand side of (5.11) satisfies [5, 6]

$$-\delta x_I{}^\mu F_{\mu\nu}{}^{(n)} = \nabla_\nu{}^{(n)} \Phi_I{}^{(n)} \tag{5.17}$$

(use (3.9a) and the definition (5.15)), which is a symmetry equation for $F_{\mu\nu}{}^{(n)}$.

As a further step we have to show that our modified set of generators $G_I{}^{(n)}$ commutes with our effective Hamiltonian of (5.7), i.e. (5.10) is valid. Using (5.11), (5.16) and (5.17) one can easily prove that

$$[G_I{}^{(n)}, \mathcal{B}_J{}^{(n)}] \doteq C_{IJ}{}^K \mathcal{B}_K{}^{(n)} \tag{5.18}$$

as a result the first term on the RHS of (5.7) commutes with $G_I{}^{(n)}$. (The proof is similar to the one usually given for the quadratic Casimir of a semi-simple Lie group, see e.g. Gilmore [16].)

Since the third term of (5.7) is a constant multiple of the unit matrix, we are only left to prove that $V^{(n)}$ commutes with our generators $G_I{}^{(n)}$. Using (5.9), (2.14), (3.3) and (3.8) and the fact that $\langle m|S_a|n\rangle$ for $m \neq n$ equals zero (see (2.13)) we get

$$\begin{aligned}(-V^{(n)})_{ij} &= \frac{1}{2M}\eta^{IJ}\mathcal{D}_I{}^{\alpha}\mathcal{D}_J{}^{\beta}E_\alpha{}^{\mu}E_\beta{}^{\nu}\langle n, i(x_0)|E_\mu{}^{\gamma}T_\gamma|m, l(x_0)\rangle\langle m, l(x_0)|E_\nu{}^{\delta}T_\delta|n, j(x_0)\rangle \\ &= \frac{1}{2M}\eta^{IJ}\mathcal{D}_I{}^{\alpha}\mathcal{D}_J{}^{\beta}\langle n, i(x_0)|T_\alpha T_\beta|n, j(x_0)\rangle\,. \end{aligned} \tag{5.19}$$

Here we also used the property $T_\alpha{}^{(n)} \equiv \langle n, i(x_0)|T_\alpha|n, j(x_0)\rangle = 0$ to allow summation for all possible values of m and l, hence obtaining the resolution of identity. Moreover, according to (4.4) the inverse matrix η^{IJ} satisfies

$$\eta^{IJ}\mathcal{D}_I{}^{K}\mathcal{D}_J{}^{L} = \eta^{KL} \tag{5.20}$$

hence

$$(-V^{(n)})_{ij} = \frac{1}{2M}\langle n, i(x_0)|\eta^{\alpha\beta}T_\alpha T_\beta|n.j(x_0)\rangle = \frac{1}{2M}\langle n, i(x_0)|\eta^{IJ}\mathcal{J}_I\mathcal{J}_J - \eta^{ab}S_aS_b|n, j(x_0)\rangle\,. \tag{5.21}$$

(The Cartan–Killing metric has a block-diagonal structure according to the decomposition (2.11), i.e. the only elements are η_{ab} and $\eta_{\alpha\beta}$ both being non-singular.) Using the formula for the second-order Casimir invariant in the representation defined by $\mathcal{U}^{(\lambda)}$

$$C(\mathcal{U}^{(\lambda)}) \equiv \eta^{IJ}\mathcal{J}_I\mathcal{J}_J \tag{5.22}$$

we obtain the result

$$(-V^{(n)})_{ij} = \frac{1}{2M}(C(\mathcal{U}^{(\lambda)}) - C(\mathcal{R}^{(n)}))\delta_{ij} \tag{5.23}$$

where $C(\mathcal{R}^{(n)})$ is the Casimir invariant of the subalgebra $\boldsymbol{h}$ in the representation defined by (2.13). Notice that for the special case $\boldsymbol{g} = su(2)$, $\boldsymbol{h} = u(1)$, $G/H \sim S^2$, $i = j = 1$ we get $V^{(n)} = (\lambda(\lambda+1) - n^2)/2M$ in agreement with [14].

Being the constant multiple of the identity $V^{(n)}$ clearly commutes with $G_I{}^{(n)}$, hence (5.10) is satisfied.

6. Examples

6.1. SO(d + 1)/SO(d)

Let us consider as our first example the rank one symmetric spaces the d- spheres, i.e. $G/H \sim SO(d+1)/SO(d) \sim S^d$. The $d(d+1)/2$ generators $\mathcal{J}_{\hat\alpha\hat\beta}$ of $SO(d+1)$ with $\hat\alpha, \hat\beta = 1, 2, \ldots, d+1$, $(\hat\alpha > \hat\beta)$, generating rotations in the $(\hat\alpha\hat\beta)$ planes satisfy the commutation relations ($d \geqslant 2$)

$$[\mathcal{J}_{\hat\alpha\hat\beta}, \mathcal{J}_{\hat\gamma\hat\epsilon}] = \delta_{\hat\alpha\hat\epsilon}\mathcal{J}_{\hat\beta\hat\gamma} + \delta_{\hat\beta\hat\gamma}\mathcal{J}_{\hat\alpha\hat\epsilon} - \delta_{\hat\alpha\hat\gamma}\mathcal{J}_{\hat\beta\hat\epsilon} - \delta_{\hat\beta\hat\epsilon}\mathcal{J}_{\hat\alpha\hat\gamma}\,. \tag{6.1}$$

We identify the generators of the subgroup $SO(d)$ as $S_{\alpha\beta} \equiv \mathcal{J}_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, d)$ $\alpha > \beta$, and the ones not belonging to the subgroup as $T_\alpha \equiv \mathcal{J}_{\alpha d+1}$, i.e we have the identification of the subalgebra indices by $a \equiv \alpha\beta$, $\alpha > \beta$. The commutation relations are

$$[S_{\alpha\beta}, S_{\gamma\epsilon}] = \delta_{\alpha\epsilon} S_{\beta\gamma} + \delta_{\beta\gamma} S_{\alpha\epsilon} - \delta_{\alpha\gamma} S_{\beta\epsilon} - \delta_{\beta\epsilon} S_{\alpha\gamma} \tag{6.2a}$$

$$[S_{\alpha\beta}, T_\gamma] = \delta_{\beta\gamma} T_\alpha - \delta_{\alpha\gamma} T_\beta \tag{6.2b}$$

$$[T_\alpha, T_\beta] = -S_{\alpha\beta} \tag{6.2c}$$

now having the (2.11) form. Reading off the structure constants needed for (4.11*a*, *b*) we can build up the matrices MN and NM having the form $(x^2 = x_\alpha x^\alpha)$

$$(MN)_\alpha{}^\beta(x) = x^2 \delta_\alpha{}^\beta - x_\alpha x^\beta \tag{6.3a}$$

$$(NM)_{(\alpha\beta)}{}^{(\gamma\epsilon)}(x) = x^\gamma x_\alpha \delta^\epsilon{}_\beta - x^\gamma x_\beta \delta^\epsilon{}_\alpha \,. \tag{6.3b}$$

Using the power-series expansions (4.14*a*, *b*) and the properties $(MN)^2 = x^2(MN)$ and $(NM)^2 = x^2(NM)$ we get

$$\cos\sqrt{MN} = I + x^{-2}(\cos x - 1)MN \tag{6.4a}$$

$$\frac{\sin\sqrt{MN}}{\sqrt{MN}} = I + x^{-2}\left(\frac{\sin x}{x} - 1\right)MN \tag{6.4b}$$

which can be used in (4.10) to build up the matrix $\mathcal{D}_I{}^J$. Similarly for the quantities (4.13*a*, *b*) and (4.8) we find

$$A_\mu{}^{\alpha\beta}(x) = \frac{\cos x - 1}{x^2} x^{[\beta} \delta_\mu{}^{\alpha]} \tag{6.5a}$$

$$E_\mu{}^\alpha(x) = \frac{\sin x}{x}\delta_\mu{}^\alpha + \left(1 - \frac{\sin x}{x}\right)\frac{x_\mu x^\alpha}{x^2} \tag{6.5b}$$

$$\Pi_\alpha{}^{\gamma\epsilon}(x) = \frac{\cos x - 1}{x \sin x} x^{[\gamma} \delta_\alpha{}^{\epsilon]} \tag{6.5c}$$

where [ ] denotes antisymmetrization of the corresponding indices.

According to (4.9) the Killing vectors are

$$\mathcal{K}_{\alpha\beta} = x_\beta \frac{\partial}{\partial x^\alpha} - x_\alpha \frac{\partial}{\partial x^\beta} \tag{6.6a}$$

$$\mathcal{K}_\alpha = x\cot x \frac{\partial}{\partial x^\alpha} + (1 - x\cot x)\frac{x_\alpha x^\beta}{x^2}\frac{\partial}{\partial x^\beta} \tag{6.6b}$$

satisfying the commutation relations (6.2*a*–*c*) with a *minus sign* (see (4.2)). Moreover using the definition (5.12) and formula (4.7*b*) we get for the compensating gauge transformation

$$W_{\alpha\beta}{}^{(n)} = S_{\alpha\beta}{}^{(n)} \tag{6.7a}$$

$$W_\alpha{}^{(n)} = \frac{1 - \cos x}{x \sin x} x^\gamma S_{\gamma\alpha}{}^{(n)} \,. \tag{6.7b}$$

Hence our modified set of $SO(d+1)$ generators is

$$G_{\alpha\beta}{}^{(n)} = -\mathcal{K}_{\alpha\beta} + W_{\alpha\beta}{}^{(n)} \tag{6.8a}$$

$$G_{\alpha}{}^{(n)} = -\mathcal{K}_{\alpha} + W_{\alpha}{}^{(n)} . \tag{6.8b}$$

Notice that our use of normal coordinates x^μ defined by (2.10) resulted in a nonlinear realization of the $SO(d+1)$ action on S^d, which when restricted to the $SO(d)$ subgroup became a linear one (see (6.6*a*, *b*)). Such realizations have been used in field theory following the paper of Coleman *et al* [17]. However, embedding S^d in $\mathbb{R}^{d+1}$ by using the coordinates

$$Y_\mu = \frac{\sin x}{x} x_\mu \qquad \mu = 1, 2 \ldots, d \tag{6.9a}$$

$$Y_{d+1} = \cos x \tag{6.9b}$$

satisfying $Y_\mu Y^\mu + Y_{d+1} Y^{d+1} = 1$, we can obtain a linear realization of the $SO(d+1)$ action, and the modified set of generators is

$$G_{\alpha\beta}{}^{(n)} = Y_\alpha \frac{\partial}{\partial Y^\beta} - Y_\beta \frac{\partial}{\partial Y^\alpha} + S_{\alpha\beta}{}^{(n)} \tag{6.10a}$$

$$G_{\alpha}{}^{(n)} = Y_\alpha \frac{\partial}{\partial Y^{d+1}} - Y_{d+1} \frac{\partial}{\partial Y^\alpha} + \frac{1}{1+Y_{d+1}} Y^\gamma S_{\gamma\alpha}{}^{(n)} . \tag{6.10b}$$

By using the coordinates (6.9*a*, *b*) we have identified $x \equiv (x_\mu x^\mu)^{1/2}$ with the polar angle θ which measures the length of the geodesic with initial tangent vector x^μ/x in accordance with the definition of normal coordinates. We stress however, that this possibility of obtaining a linear realization from a nonlinear one crucially depends on the embedding of G/H in $\mathbb{R}^d$ for d conveniently chosen.

To be more specific let us consider a class of models where the slow configuration spaces are *even*-dimensional spheres [15, 18]. According to (2.2) the kinetic term is the second-order Laplace–Beltrami operator on S^{2l} ($d = 2l$). Let us specify $\mathcal{U}^{(\lambda)}$ in (2.2) to be the $(2^l \times 2^l)$-dimensional spinor representation of $SO(2l+1)$, which, when restricted to the $SO(2l)$ subgroup, is reducible and contains the *two* $(2^{l-1} \times 2^{l-1})$-dimensional spinor representations of $SO(2l)$, with positive and negative chirality. Let H_0 be the matrix

$$H_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{6.11}$$

(I is the $2^{l-1} \times 2^{l-1}$ unit matrix) clearly commuting with the restriction of $\mathcal{U}^{(\lambda)}$ to the subgroup $SO(2l)$ which is block diagonal. Recall that the spinor representation of $SO(2l+1)$ is generated by a $(2l+1)$-dimensional Clifford algebra with generators $\Gamma_{\hat\alpha}$ $(\hat\alpha = 1, 2, \ldots, 2l+1)$ satisfying

$$\{\Gamma_{\hat\alpha}, \Gamma_{\hat\beta}\} = 2\delta_{\hat\alpha\hat\beta}\mathbf{1} \tag{6.12}$$

where $\mathbf{1}$ is the $2^l \times 2^l$ unit matrix. The generators of $so(2l+1)$ in this representation are

$$S_{\alpha\beta} = \tfrac{1}{4}[\Gamma_\alpha, \Gamma_\beta] \qquad \alpha, \beta = 1, 2, \ldots, 2l \tag{6.13a}$$

$$T_\alpha = \tfrac{1}{4}[\Gamma_\alpha, \Gamma_{2l+1}] \tag{6.13b}$$

satisfying (6.2a–c). We can use, for example, the representation

$$\Gamma_\alpha \equiv \begin{pmatrix} 0 & -\mathrm{i}\Sigma_\alpha{}^\dagger \\ \mathrm{i}\Sigma_\alpha & 0 \end{pmatrix} \qquad \Gamma_{2l+1} \equiv H_0 \qquad \alpha = 1, 2, \ldots, 2l \tag{6.14}$$

where the Σ_α are $2^{l-1} \times 2^{l-1}$ matrices satisfying $\{\Sigma_\alpha, \Sigma_\beta{}^\dagger\} = 2\delta_{\alpha\beta} I$. (The explicit form of these matrices can be given using a recursive procedure starting from the usual Pauli matrices, see [15, 18].) According to (6.11) we have two 2^{l-1}-fold degenerate subspaces with corresponding eigenvalues ± 1, hence we can label the eigensubspaces by $n \equiv (\pm)$. Moreover one can easily prove using (2.10) for $\mathcal{U}^{(\lambda)}(x)$ with (6.13b), and (6.11) in (2.2) that

$$H(Y(x)) = \Gamma_{\hat{\alpha}} Y_{\hat{\alpha}}(x) \qquad \hat{\alpha} = 1, 2, \ldots, 2l+1\,. \tag{6.15}$$

where (6.9a, b) were used. Introducing $P_{\hat{\alpha}} \equiv -\mathrm{i}\partial/\partial Y^{\hat{\alpha}}$ our class of models in (2.2) can be expressed in the simple form

$$H_{\mathrm{tot}} = \frac{1}{2M} P^2 + \Gamma_{\hat{\mu}} Y_{\hat{\mu}} \qquad \hat{\mu} = 1, 2, \ldots, 2l+1\,. \tag{6.16}$$

By calculating the quantities $\mathcal{A}_{\hat{\mu}}{}^{(n)}$, $V^{(n)}$ of (5.7) we can build up the effective Hamiltonians $H_{\mathrm{eff}}{}^{(n)}$ (see [15] for a detailed discussion). Of course the modified generators defined by (6.10a, b) with

$$S_{\alpha\beta}{}^{(+)} = \tfrac{1}{4}[\Sigma_\alpha{}^\dagger, \Sigma_\beta] \tag{6.17a}$$

$$S_{\alpha\beta}{}^{(-)} = \tfrac{1}{4}[\Sigma_\alpha, \Sigma_\beta{}^\dagger] \tag{6.17b}$$

will commute with this effective Hamiltonian. Notice also that the matrix elements $T_\alpha{}^{(+)}$ and $T_\alpha{}^{(-)}$ are zero satisfying our restriction imposed on our class of models.

Let us consider now the special case $l = 1$ which is just the example of section 2, (see (2.3)–(2.9)). The coset is $SO(3)/SO(2) \sim SU(2)/U(1) \sim S^2$ the 2-sphere. Since the geometrical properties of our coset characterized by the quantities $A_\mu{}^a$, $E_\mu{}^\alpha$, $\mathcal{K}_l$ are independent of the particular representation, we can use (6.5a, b) and (6.6a, b) to calculate them, and then the ones of (6.7) and (6.8) depending on the representation through the matrix elements $S_a{}^{(n)}$. Let $\mathcal{U}^{(\lambda)}$ be the usual $SU(2)$ representation characterized by the (integer or half-integer) number λ, and the matrices of (2.5) are also supposed to be in this representation. Let us choose $H_0 \equiv \mathrm{i}\mathcal{J}_3$. The eigenvalues of H_0 are restricted by $-\lambda \leqslant n \leqslant \lambda$. The subalgebra $\boldsymbol{h}$ is spanned by $S \equiv \mathcal{J}_3$ alone ($a = 1$). Moreover we have

$$S^{(n)} = -\mathrm{i}n \tag{6.18}$$

(recall our convention of using anti-Hermitian generators). Using the coordinates (2.3) we can easily reduce formulae (6.5)–(6.8) for our case. However it is more instructive to use the Cartesian coordinates (X_1, X_2, X_3) (see (2.7)). For the gauge-potential $\mathcal{A}_{\hat{\mu}}{}^{(n)}$ ($\hat{\mu} = 1, 2, 3$) we get

$$\mathcal{A}_{\hat{\mu}}{}^{(n)}(X) = -\mathrm{i}n \frac{1}{1 + X_3} \begin{pmatrix} X_2 \\ -X_1 \\ 0 \end{pmatrix} \qquad \hat{\mu} = 1, 2, 3 \tag{6.19}$$

i.e. the vector potential of a monopole (well defined on the northern hemisphere) with monopole strength n. Straightforward calculation yields the Killing vectors of (2.8) and the compensating gauge transformation

$$W_I{}^{(n)}(X) = -\mathrm{i}n \begin{pmatrix} X_1/(1+X_3) \\ X_2/(1+X_3) \\ 1 \end{pmatrix} \qquad I = 1, 2, 3 . \tag{6.20}$$

The modified generators according to (6.8a, b) are

$$G_I{}^{(n)} = -\varepsilon_{I\hat{\mu}\hat{\nu}} X_{\hat{\mu}} \frac{\partial}{\partial X^{\hat{\nu}}} + W_I{}^{(n)} \tag{6.21}$$

where the indices of $\hat{\mu}$, $\hat{\nu}$, and I have the same range. We can obtain a more familiar form by using the quantity $\Phi_I{}^{(n)}$ of (5.15). Since $\delta X_I{}^{\hat{\mu}}(X) \equiv \varepsilon_{I\hat{\nu}\hat{\mu}} X_{\hat{\nu}}$ we get, from (5.15),

$$\Phi_I{}^{(n)} = -\mathrm{i}n X_{\hat{\mu}} \delta^{\hat{\mu}}{}_I \qquad I = 1, 2, 3 . \tag{6.22}$$

By virtue of (5.8) and (5.16) the alternative form is

$$G_I{}^{(n)} = -\varepsilon_{I\hat{\mu}\hat{\nu}} X_{\hat{\mu}} (\partial_{\hat{\nu}} + \mathcal{A}_{\hat{\nu}}{}^{(n)}) - \mathrm{i}n X_{\hat{\nu}} \delta^{\hat{\nu}}{}_I \tag{6.23}$$

which is the well known modification of angular momentum operators when magnetic monopoles are present. The usual form of these operators [5, 6, 19] involves the quantity $R \equiv (X_{\hat{\mu}} X^{\hat{\mu}})^{1/2}$ which is not equal to 1 as in our case. Then we can regard the gauge field (6.19) living in $\mathbf{R}^3 - \{0\}$ rather on S^2. In this case $(1 + X_3)$ in (6.19) have to be replaced by $R(R + X_3)$, and $-\mathrm{i}nX_{\hat{\nu}}$ in (6.23) by $-\mathrm{i}nX_{\hat{\nu}}/R$.

Similarly the gauge-fields obtained from (6.5a) using (3.11) and (6.17a,b) can be extended from S^{2l} to $\mathbf{R}^{2l+1} - \{0\}$ by introducing $R \equiv (Y_{\hat{\mu}} Y^{\hat{\mu}})^{1/2}$ ($\hat{\mu} = 1, 2, \ldots, 2l+1$) into our formulae. These gauge-fields having the form

$$\mathcal{A}^{(\pm)} = \frac{1}{R(R + Y_{2l+1})} Y^{\mu} \,\mathrm{d}Y^{\nu} S_{\mu\nu}{}^{(\pm)} \qquad \mu, \nu = 1, 2, \ldots, 2l \tag{6.24}$$

are the higher-dimensional non-Abelian monopoles studied in [20]. A straightforward calculation shows that in this case our modified generators (6.10a, b) can be cast into the form

$$G_{\hat{\mu}\hat{\nu}}{}^{(\pm)} = Y_{\hat{\mu}}(\partial_{\hat{\nu}} + \mathcal{A}_{\hat{\nu}}{}^{(\pm)}) - Y_{\hat{\nu}}(\partial_{\hat{\mu}} + \mathcal{A}_{\hat{\mu}}{}^{(\pm)}) + R^2 F_{\hat{\mu}\hat{\nu}}{}^{(\pm)} \tag{6.25}$$

where $F_{\hat{\mu}\hat{\nu}}{}^{(\pm)} = \partial_{\hat{\mu}} \mathcal{A}_{\hat{\nu}}{}^{(\pm)} - \partial_{\hat{\nu}} \mathcal{A}_{\hat{\mu}}{}^{(\pm)} + [\mathcal{A}_{\hat{\mu}}{}^{(\pm)}, \mathcal{A}_{\hat{\nu}}{}^{(\pm)}]$ with $\mathcal{A}_{2l+1}{}^{(\pm)} = 0$. To prove (6.25) notice that it is of the (6.23) form, and then we only have to show that $R^2 F_{\hat{\mu}\hat{\nu}}{}^{(\pm)} = \Phi_{\hat{\mu}\hat{\nu}}{}^{(\pm)}$ where $\Phi_{\hat{\mu}\hat{\nu}}{}^{(\pm)}$ is the antisymmetric $(2l+1) \times (2l+1)$ matrix built up from the components $\Phi_a{}^{(\pm)} \equiv \Phi_{\mu\nu}{}^{(\pm)}$, $\mu\nu = 1, 2, \ldots, 2l$, and $\Phi_{\mu}{}^{(\pm)}$ of (5.15). The set of modified generators in (6.25) can be regarded as a higher-dimensional generalization of the one studied by Yang [9] in the special case when $l = 2$, in order to obtain the $SU(2)$ monopole harmonics.

6.2. SO(d,1)/SO(d) and algebraic scattering theory

It is very easy to generalize these results for the coset space $SO(d,1)/SO(d)$, i.e. by choosing for G the non-compact group $SO(d,1)$ and for H its maximal compact subgroup $SO(d)$. The resulting coset space is the upper sheet of the double-sheeted hyperboloid, that can be imbedded in $\mathbf{R}^{d,1}$ by using instead of (6.9a, b) the coordinates

$$Z_\mu = \frac{\sinh x}{x} x_\mu \tag{6.26a}$$

$$Z_{d+1} = \cosh x \tag{6.26b}$$

satisfying $-Z_\mu Z^\mu + Z_{d+1}Z^{d+1} = 1$. In this case the modified generators are

$$G_{\alpha\beta}{}^{(n)} = Z_\alpha \partial_\beta - Z_\beta \partial_\alpha + S_{\alpha\beta}{}^{(n)} \tag{6.27a}$$

$$G_\alpha{}^{(n)} = Z_\alpha \partial_{d+1} + Z_{d+1}\partial_\alpha + \frac{1}{1+Z_{d+1}} Z^\gamma S_{\gamma\alpha}{}^{(n)} \tag{6.27b}$$

satisfying the commutation relations of the $so(d,1)$ Lie algebra, and the matrices $S_{\alpha\beta}{}^{(n)}$ form a representation of $so(d)$. (Notice the sign change in (6.27b) and recall that for $so(d,1)$ the commutation relations of (6.2a–c) are the same except for the *minus sign* of $S_{\alpha\beta}$ in (6.2c) which has to be changed.) In this way we have obtained a non-standard realization of the Lie algebra of the *non-compact group* $SO(d,1)$. An important property of $SO(d,1)$ is that among its infinite-dimensional unireps there is one indexed by a *continuous series* of numbers [21]

$$j = -\tfrac{1}{2}(d-1) + \mathrm{i}k \tag{6.28}$$

where $k > 0$ is a real number.

According to the idea of algebraic scattering theory [22] such representations can be used to characterize scattering states of a Hamiltonian written as a function of one of the Casimir operators C of G. Some authors [23] also stressed the physical relevance of obtaining explicit interaction terms by using particular coordinate realizations for C coming from realizations of $\boldsymbol{g}$.

In this spirit we can try to use our realization of the $so(d,1)$ algebra given by (6.27a, b) to build up the Casimir operator $C^{(n)} = -\frac{1}{2}G_{\alpha\beta}{}^{(n)}G^{\alpha\beta(n)} + G_\alpha{}^{(n)}G^{\alpha(n)}$. The eigenvalue problem for $C^{(n)}$ is [21]

$$C^{(n)}|k,m,n\rangle = \left(-\left(\frac{d-1}{2}\right)^2 - k^2\right)|k,m,n\rangle \tag{6.29}$$

where

$$\tfrac{1}{2}G_{\alpha\beta}{}^{(n)}G^{\alpha\beta(n)}|k,m,n\rangle = m(2-d-m)|k,m,n\rangle \tag{6.30}$$

i.e. the eigenvalue problem for the second-order Casimir of the maximal compact subgroup $SO(d)$. Equation (6.29) can be regarded as a Schrödinger equation describing a scattering process with scattering energy k^2. Notice also that there is another $SO(d)$ subgroup involved in this process namely the one described by the generators $S_{\alpha\beta}{}^{(n)}$ appearing in (6.27a, b). It is interesting to note that the presence of this subgroup can be regarded as the hallmark of some *fast variables* according to section 5. Moreover we expect that both sets of quantum numbers m and n will appear in the interaction term obtained from (6.26). An explicit calculation was carried out for the simplest case, namely $G \sim SO(2,1)$ $H \sim SO(2) \sim U(1)$ $(d=2)$ [24]. In this case (6.29) yields [24]

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + V_{m,n}(\rho)\right) f_{k,m,n}(\rho) = k^2 f_{k,m,n}(\rho) \tag{6.31}$$

where $\rho \equiv (x_\mu x^\mu)^{1/2}$, $\mu = 1, 2$, and

$$V_{m,n}(\rho) = \frac{(n+m)^2 - \frac{1}{4}}{4\sinh^2(\rho/2)} - \frac{(n-m)^2 - \frac{1}{4}}{4\cosh^2(\rho/2)} \,. \tag{6.32}$$

After solving (6.31), we can determine the S-matrix from the asymptotic form of the solution. The result is in agreement with the one obtained by using the algebraic scattering theory.

Calculating the quantities of section 4 and using the expression for the modified generators (5.14), we can obtain realizations of other non-compact groups, on symmetric spaces. Higher rank symmetric spaces, e.g. $SO(p,q)/SO(p) \times SO(q)$ where $\min(p,q) > 1$, are especially interesting since they can describe higher-dimensional scattering. A well studied example of that kind is the group $SO(3,2)$. This group can be useful for the study of heavy-ion reactions [25]. Since the very construction of our modified generators clearly shows the possible connection with the dynamics of two different types of variables, then two types of quantum numbers will appear in the expression for the S-matrix. From the range of such quantum numbers in the scattering matrix we can gain some information on the form of G representations characterizing the dynamics of the fast (internal) degrees of freedom. Hence, besides the possibility of describing the scattering data by employing a non-compact group G, we also have the advantage to implement this symmetry group in some dynamics compatible with such data. Therefore such realizations might deserve some attention in physical applications.

7. Modified symmetry generators and induced representations

Having explained the origin of the modified generators, let us turn to the question of what kind of role they are playing. According to (5.10) such generators will yield the 'good' quantum numbers for the effective theory. Moreover, according to (5.10) we expect that $H_{\mathrm{eff}}{}^{(n)}$ can be expressed entirely in terms of the Casimir operators of G and H in the particular representations used for them. Since the term $V^{(n)}$ of (5.7) can indeed be represented in this form (see (5.23)), we hope that we can do this for the first term on the right-hand side of (5.7) too. To show that this is really the case we compute $\eta^{IJ} G_I{}^{(n)} G_J{}^{(n)}$ by using (5.16). Comparing the definition of $\Phi_I{}^{(n)}$ (5.15), using (5.12), with (4.7b) we obtain the following expression for $\Phi_I{}^{(n)}$:

$$\Phi_I{}^{(n)} = \mathcal{D}_I{}^a(x) S_a{}^{(n)} \,. \tag{7.1}$$

Notice also that according to our restriction $T_\alpha{}^{(n)} = 0$ this quantity is simply $\langle n, i(x)|\mathcal{J}_I|n, j(x)\rangle$. Using (4.7$a$) and (5.8)

$$\eta^{IJ} \Phi_I{}^{(n)} \mathcal{B}_J{}^{(n)} = -\eta^{IJ} \mathcal{D}_I{}^a \mathcal{D}_J{}^\alpha E_\alpha{}^\mu S_a{}^{(n)} \nabla_\mu{}^{(n)} = 0 \tag{7.2}$$

according to (5.20) and the block-diagonal structure of η^{IJ}. Moreover by virtue of (5.8) and (5.17) we have

$$\eta^{IJ}(\mathcal{B}_I{}^{(n)}\Phi_J{}^{(n)}) = -\eta^{IJ}\delta x_I{}^{\mu}\nabla_{\mu}{}^{(n)}\Phi_J{}^{(n)} = \eta^{IJ}\delta x_I{}^{\mu}\delta x_J{}^{\nu}F_{\nu\mu} = 0. \tag{7.3}$$

Now we can calculate $\eta^{IJ}G_I{}^{(n)}G_J{}^{(n)}$ using (5.16) and (7.1)

$$\mathcal{C}(\mathcal{G}^{(n)}) \equiv \eta^{IJ}G_I{}^{(n)}G_J{}^{(n)} = \eta^{IJ}\mathcal{B}_I{}^{(n)}\mathcal{B}_J{}^{(n)} + \eta^{IJ}\mathcal{D}_I{}^{a}\mathcal{D}_J{}^{b}S_a{}^{(n)}S_b{}^{(n)}. \tag{7.4}$$

Using again (5.20) we get the result for the first term on the right-hand side of (5.7) as

$$\frac{1}{2M}\eta^{IJ}\mathcal{B}_I{}^{(n)}\mathcal{B}_J{}^{(n)} = \frac{1}{2M}(\mathcal{C}(\mathcal{G}^{(n)}) - \mathcal{C}(\mathcal{R}^{(n)})) \tag{7.5}$$

where $\mathcal{C}(\mathcal{R}^{(n)})$ is the quadratic Casimir operator of h in the representation defined by (2.13). Hence the final form of the effective Hamiltonian is

$$H_{\text{eff}}{}^{(n)} = -\frac{1}{2M}\left(\mathcal{C}(\mathcal{G}^{(n)}) + \mathcal{C}(\mathcal{U}^{(\lambda)}) - 2\mathcal{C}(\mathcal{R}^{(n)})\right) + \varepsilon^{(n)}. \tag{7.6}$$

Since the representation content of the fast Hamiltonian $H(x)$ of (2.2) is fixed, we have fixed values for the Casimir invariants of $\mathcal{C}(\mathcal{U}^{(\lambda)})$ and $\mathcal{C}(\mathcal{R}^{(n)})$. Hence the problem we have to solve is to find the eigenvalues and eigenfunctions of $\mathcal{C}(\mathcal{G}^{(n)})$. Notice that the functions we are searching for are actually square-integrable sections of the homogeneous vector bundle with base space G/H and fibre a $\dim\mathcal{R}^{(n)}$-dimensional complex vector space.

In order to clarify these issues we have to go back to the Born–Oppenheimer expansion of the total wavefunction $|\Psi\rangle$ satisfying $(H_{\text{tot}} - E)|\Psi\rangle = 0$. Moreover, in the following we assume that G is compact. Then we can choose a finite-dimensional unitary irreducible representation for $\mathcal{U}^{(\lambda)}$. Since we now have a finite-dimensional Hilbert space for our fast degrees of freedom we define

$$\Psi_p(x) \equiv (\langle x| \otimes \langle p|)|\Psi\rangle \tag{7.7}$$

where $|p\rangle$, $p = 1, 2, \ldots, \dim\mathcal{U}^{(\lambda)}$ are basis vectors for this representation. The action of an unitary representation $U(g)$ of G on $\Psi_p(x)$ is

$$\Psi_p{}'(x) \equiv (U(g)\Psi)_p\,(x) = \mathcal{U}^{(\lambda)}{}_{qp}(g)\Psi_q(g^{-1}x) \tag{7.8}$$

where $\mathcal{U}^{(\lambda)}{}_{pq}(g)$ is the matrix of the unitary irrep $\mathcal{U}^{(\lambda)}(g)$ in the basis $|p\rangle$, whose restriction to the subgroup H contains the unitary irrep $\mathcal{R}^{(n)}$ of H. Introducing the $|p\rangle$ representation for the eigenstates of the fast Hamiltonian as

$$\varphi_{pi}{}^{(n)}(x) \equiv \langle p|n, i(x)\rangle \qquad p = 1, 2, \ldots, \dim\mathcal{U}^{(\lambda)} \quad i = 1, 2, \ldots, \dim\mathcal{R}^{(n)} \tag{7.9}$$

we can express the expansion of the total wavefunction $\Psi_p(x)$ in the form

$$\Psi_p(x) = \sum_{n,i}\varphi_{pi}{}^{(n)}(x)\psi_i{}^{(n)}(x). \tag{7.10}$$

Note that in this expansion the $\varphi_{pi}{}^{(n)}$ are playing the role of 'basis vectors' for the degenerate eigensubspace and the 'expansion coefficients' $\psi_i{}^{(n)}$ are satisfying the Schrödinger equation with $H_{\text{eff}}{}^{(n)}$ expressed as in (5.7).

Let us now compute $\Psi_p{}'(gx)$ using the expansion (7.10). The expansion of $\Psi_p{}'(gx)$ will be of the same form (with x replaced by gx) as in (7.10) except for some transformation of the 'coefficients' $\psi_i{}^{(n)'}(gx)$. Using (7.9) and (2.17) we obtain the result

$$\psi_j{}^{(n)'}(gx) = \mathcal{R}_{ji}{}^{(n)}(h(g,x))\psi_i{}^{(n)}(x)\,. \tag{7.11}$$

Hence the representation of G on the sections $\psi_j{}^{(n)}(x)$ of our homogeneous eigensubspace bundle is the representation induced by the irrep $\mathcal{R}^{(n)}$ of the subgroup H. (For a nice and short review of vector bundles and induced representations see e.g. [26, appendix B].) It can be shown that the transformation property (7.11) defines a *unitary* representation $U^{\mathcal{R}}$ of G on the space of square-integrable sections $\psi_j{}^{(n)}(x)$. Using the infinitesimal version of (7.11) by employing $gx \sim x + u^I \delta x_I$, and $h(g,x) \sim I + u^I W_I$ one can prove that the modified generators $G_I{}^{(n)}$ are nothing more then the infinitesimal generators of this unitary representation induced by the irrep $\mathcal{R}^{(n)}$ of the subgroup H. More explicitly we have

$$\left(U^{\mathcal{R}}(\mathcal{J}_I)\psi\right)_j{}^{(n)}(x) = (-\delta x_I{}^{\mu}\partial_\mu + W_I{}^{(n)})_{ij}\psi_i{}^{(n)}(x)\,. \tag{7.12}$$

Having clarified the group-theoretical meaning of the generators $G_I{}^{(n)}$ we are now in a position to say something about the allowed values for the Casimir invariants of $C(\mathcal{G}^{(n)})$ in (7.6). First recall that the induced representation can be written as a direct sum of irreps of G. We would like to know what kind of finite-dimensional unitary irreps appear in the decomposition. According to the Frobenius reciprocity theorem [26, 27] the multiplicity of a particular finite-dimensional irrep in the induced representation is equal to the multiplicity of our fixed representation $\mathcal{R}^{(n)}$ of the subgroup H in the restriction of this finite-dimensional irrep of G to H. Let us denote this finite-dimensional unitary irrep of G by $\mathcal{U}^{(\Lambda)}$. In particular the representation $\mathcal{U}^{(\lambda)}$ used to fix the representation content of our fast variables (see (2.2)) corresponds to one special value for (Λ) labelling this representation. Since this representation contains $\mathcal{R}^{(n)}$ it will also be contained in the induced representation, with the same multiplicity. More generally the allowed representations $\mathcal{U}^{(\Lambda)}$ in the induced representation are the ones containing $\mathcal{R}^{(n)}$ when restricting them to the subgroup H. Hence the space of vector-valued functions $\psi_j{}^{(n)}(x)$ on G/H (sections) can be decomposed into a direct sum of 'allowed' invariant subspaces characterized by eigenvalues for the Casimir operator $C(\mathcal{G}^{(n)})$ on them.

As an example let us consider the simple model discussed in section 2 (2.3)–(2.9). For $\mathcal{U}^{(\lambda)}$ we take an $SU(2)$ representation characterized by a *fixed* λ. For the subgroup irrep $\mathcal{R}^{(n)}$ we take one of the (one-dimensional) representations contained in the restriction of $\mathcal{U}^{(\lambda)}$ to $H \sim U(1)$ (generated by $\mathcal{J}_3$ alone) characterized by the number n. Then in the induced representation only those $\mathcal{U}^{(\Lambda)}$ will appear for which $\Lambda = |n|, |n+1|, \ldots$. In particular when n is half-integer only half-integer Λ will appear. Hence the eigenvalues of $H_{\mathrm{eff}}{}^{(n)}$ on the allowed subspaces are

$$E^{(\lambda,n)}{}_{\Lambda} = \frac{1}{2M}(\Lambda(\Lambda+1) + \lambda(\lambda+1) - 2n^2) + n \qquad \Lambda = |n|, |n+1|, \ldots\,. \tag{7.13}$$

Notice, that for the total Hamiltonian H_{tot} slow rotations have been generated by the standard angular momentum operators (2.8) with *integer* Λ. However, after integrating out the fast variables we obtained an $H_{\mathrm{eff}}{}^{(n)}$ with modified angular momentum operators with the possibility for Λ also being *half-integer*.

How can we describe the eigensections of $H_{\mathrm{eff}}{}^{(n)}$ of (7.6)? In other words we are searching for the components of the square-integrable eigensections that are irreducible under G, i.e. the 'harmonics' of the fields. The answer is well known in the mathematics literature in connection with harmonic expansions of arbitrary fields defined on compact homogeneous spaces. We merely refer to the result [26] according to which ${\psi_i}^{(n)}(x)$ can be expanded as follows

$$
{\psi_i}^{(n)}(x) = \sum_{\Lambda}\sum_{p,\xi} {\psi_{\Lambda\xi p}}^{(n)}\mathcal{U}^{(\Lambda)}(L^{-1}(x))_{i\xi p} \tag{7.14}
$$

where $p = 1, 2, \ldots, \dim\mathcal{U}^{(\Lambda)}$, $i = 1, 2, \ldots, \dim\mathcal{R}^{(n)}$, and summation is restricted only to the 'allowed' values of Λ, and ξ is the multiplicity of the representation $\mathcal{R}^{(n)}$ of H in the restriction of $\mathcal{U}^{(\Lambda)}$ to H. Equation (7.14) tells us that the harmonics are essentially matrix elements of the irreps Λ for which $\xi \neq 0$.

For our example the sum over representations in (7.14) is restricted to the values $\Lambda = |n|, |n+1|, \ldots$, where n is fixed by the choice of eigensubspace for forming ${H_{\text{eff}}}^{(n)}$. The harmonics are the Wigner D-functions ${D_{np}}^{(\Lambda)}(L^{-1}(x))$ $p = -\Lambda, \ldots, \Lambda$, also called the monopole harmonics [19].

For the general case, in order to determine the decomposition of the space of sections we have to know the multiplicities of $\mathcal{R}^{(n)}$ for any $\mathcal{U}^{(\Lambda)}$. This task has to be carried out for the given fields ${\psi_i}^{(n)}$ in question. For example, in case of the even-dimensional spheres of section 6, ${\psi_i}^{(n)}$ $i = 1, 2, \ldots, 2^{l-1}$ are spinor fields on S^{2l} transforming with respect to Spin$(2l)$ the two-fold covering of $SO(2l)$. Hence to find the eigenvalues of the Casimir operator $\eta^{IJ}{G_I}^{(n)}{G_J}^{(n)}$ we have to diagonalize the corresponding Casimir operator of Spin$(2l+1)$, and keeping only those eigenvalues which correspond to representations of Spin$(2l+1)$ containing the spinor representation of Spin$(2l)$ with positive ($\mathcal{R}^{(+)}$) or negative ($\mathcal{R}^{(-)}$) chirality.

Finally we give a brief sketch of the $l = 2$ case, which can be regarded as the physical example of a spin $\frac{3}{2}$ particle interacting with quadrupole electric field [28]. Moreover this appears to be the simplest case involving non-trivial non-Abelian gauge-fields (instantons). Here the coset is the 4-sphere $S^4 \sim$ Spin(5)/Spin(4) $\sim SO(5)/SO(4)$. The modified generators of $g \sim$ spin(5) $\sim so(5)$ given by (6.25) with $\hat{\mu}, \hat{\nu} = 1, \ldots, 5$ act on the square-integrable sections of the homogeneous vector bundle over S^4, where the fibrum is now two dimensional. $\mathcal{U}^{(\lambda)}$ is the four-dimensional spinor representation of $SO(5)$, and the $SO(4)$ representation contained in its restriction is simply the block-diagonal part of this 4×4 matrix, containing two $SU(2)$ blocks corresponding to the positive (negative) chirality spinor representations of $SO(4) \sim (SU(2) \times SU(2))/Z_2$. Since the group $SO(5)$ is rank two we can label its irreps by the two integers a_1 and a_2, $a_1 \geqslant a_2 \geqslant 0$. Following [9]. we will refer to a particular irrep of $SO(5)$ by $(a_1, a_2)_5$. Since $SO(4)$ has the product structure of two $SU(2)$ we can label its irreps by the two numbers b_1 and b_2, where $b_1 = 0, \frac{1}{2}, 1, \ldots, b_2 = 0, \frac{1}{2}, 1 \ldots$, in the form $(b_1, b_2)_4$. In this notation [9]

$$
C(\mathcal{U}^{|a_1,a_2|}) = -\tfrac{1}{2}(a_1(a_1+4) + a_2(a_2+2))I_{\mathcal{N}} \tag{7.15}
$$

where

$$
\mathcal{N}(\mathcal{U}^{|a_1,a_2|}) = (1 + \tfrac{1}{2}a_1)(1 + a_2)\big(1 + \tfrac{1}{3}(a_1 + a_2)\big)(1 + a_1 - a_2) \tag{7.16}
$$

and $I_{\mathcal{N}}$ is the $\mathcal{N}$-dimensional unit matrix. In particular the four-dimensional spinor irrep of $SO(5)$ is $(1, 0)_5$, with Casimir invariant $-\frac{5}{2}$. The representations $\mathcal{R}^{(+)}$ and $\mathcal{R}^{(-)}$ are the ones with $(\frac{1}{2}, 0)_4$ and $(0, \frac{1}{2})_4$. If we choose, e.g. the eigensubspace with positive energy (chirality), we have to find the representations $(a_1, a_2)_5$ which contain the irrep $(\frac{1}{2}, 0)_4$. Using the result [9]

$$(a_1, a_2)_5 = \sum \left(\frac{r+s}{2}, \frac{r-s}{2} \right)_4 \tag{7.17}$$

where $r = \frac{1}{2}(a_1 - a_2), \frac{1}{2}(a_1 - a_2) + 1, \ldots, \frac{1}{2}(a_1 + a_2)$ and $s = \frac{1}{2}(a_2 - a_1), \frac{1}{2}(a_2 - a_1) + 1, \ldots, \frac{1}{2}(a_1 - a_2)$, we can easily show that these representations have the form $(a_1, a_1 - 1)_5$. In particular the irrep $(1, 0)_5$ clearly contains $(\frac{1}{2}, 0)_4$. Hence according to the Frobenius reciprocity theorem the irreps $(a_1, a_1 - 1)_5$ are contained in the induced representation. The final result for $\mathcal{C}(\mathcal{G}^{(\pm)})$ is

$$\mathcal{C}(\mathcal{G}^{(\pm)}) = \sum_{a_1 > 0} \left(-(a_1 + 1)^2 + \tfrac{3}{2} \right) I_{\mathcal{N}(a_1)} \tag{7.18}$$

where

$$\mathcal{N}(a_1) = \tfrac{4}{3} a_1 (a_1 + 1)\left(\tfrac{1}{2} a_1 + 1\right). \tag{7.19}$$

This is the same result Yang obtained studying the indicial equation of the differential equation corresponding to the eigenvalue problem of $\mathcal{C}(\mathcal{G}^{(\pm)})$. Using these results and (7.6) we can easily write down the allowed set of eigenvalues of ${H_{\text{eff}}}^{(\pm)}$.

8. Conclusions

In this paper we have investigated the origin of modified symmetry generators by employing a simple class of models containing two types of variables: 'slow' and 'fast'. The total system, containing both types of degrees of freedom was supposed to have a symmetry characterized by the semi-simple Lie group G, meaning that simultaneous G rotations of both types of variables leave the system invariant. As a further step we have taken into account the different energy scales associated with our different types of variables by using the Born–Oppenheimer method. This procedure was based on the assumption that the energy spacings between the energy levels corresponding to *degenerate* eigensubspaces, are sufficiently large, hence no transitions due to the slow system can occur between them. Restricting our attention to one particular eigensubspace labelled by n, yielded an effective Hamiltonian ${H_{\text{eff}}}^{(n)}$ for the slow system. Due to the presence of induced non-Abelian gauge fields in ${H_{\text{eff}}}^{(n)}$, the G rotations on the slow degrees of freedom are realized in a non-trivial way. The generators of such rotations are the modified generators ${G_I}^{(n)}$, $I = 1, 2, \ldots, \dim g$, corresponding to our particular choice of a degenerate eigensubspace.

We have given illustrative examples for coset space models based on spheres and hyperboloids. We identified the modified symmetry generators as the generators of the induced representation of G, induced by a subgroup representation of H coming from the fast variables, acting on the sections of the eigensubspace bundle over G/H. This result enabled us to recast the problem of exotic quantum numbers for effective quantum systems in purely algebraic terms via the Frobenius reciprocity theorem. Moreover we have shown that both the Born–Oppenheimer scalar potential (5.23) and the kinetic term containing induced gauge fields (7.5) can be expressed in terms of the Casimir invariants of G and H.

The notion that for homogeneous spaces G/H differential relations reduce to algebraic ones involving matrix representations of G and H, has already been extensively used by physicists following the influential work of Salam and Strathdee [29]. In this case harmonic analysis on coset spaces provided a means for calculating the excitation spectra for dimensionally-reduced (effective) theories, originally formulated in higher dimensions. It is interesting then that such techniques now find some application in the study of coupled systems of two types of degrees of freedom with symmetry.

Acknowledgments

The author would like to express his gratitude to Professor M V Berry for the warm hospitality at the theory group of the H H Wills Physics Laboratory, Bristol. This work has been supported by a Royal Society postdoctoral fellowship awarded in 1992, and by the OTKA under grant nos 517-518/1990, T7283/1993, and DFG/MTA under contract no 45/1992.

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